Tutorial Notes 3

1. (a) Write $|\exp(2z+i)|$ and $|\exp(iz^2)|$ in terms of x and y. Then show that

$$|\exp(2z+i)| + |\exp(iz^2)| \leq e^{2x} + e^{-2xy}.$$

(b) Show that $|\exp(z^2)| \leq \exp(|z|^2)$.
(c) Prove that $|\exp(-2z)| < 1$ if and only if $\operatorname{Re} z > 0$.

2. Show that $\overline{\exp(iz)} = \exp(i\bar{z})$ if and only if $z = n\pi \quad (n = 0, \pm 1, \pm 2, \ldots)$.

3. (a) Recall (Sec. 6) that if $z = x + iy$, then:

$$x = \frac{z+\bar{z}}{2}, \quad y = \frac{z-\bar{z}}{2i}.$$

By formally applying the chain rule in calculus to a function $F(x,y)$ of two real variables, derive the expression:

$$\frac{\partial F}{\partial \bar{z}} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial F}{\partial x} + i\frac{\partial F}{\partial y}\right).$$

(b) Define the operator:

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right),$$

suggested by part (a), to show that if the first-order partial derivatives of the real and imaginary components of a function $f(z) = u(x,y) + iv(x,y)$ satisfy the Cauchy-Riemann equations, then:

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left[(u_x - v_y) + i(v_x + u_y)\right] = 0.$$

Thus derive the *complex form* $\frac{\partial f}{\partial \bar{z}} = 0$ of the Cauchy-Riemann equations.

Problem 1(a): Express $|\exp(2z+i)|$ and $|\exp(iz^2)|$ in terms of x and y, and show that

$$|\exp(2z+i)| + |\exp(iz^2)| \leq e^{2x} + e^{-2xy}.$$

Solution:
For any complex number $z = x + iy$,

$$\exp(z) = e^x e^{iy}.$$

Since $|e^{iy}| = 1$, we get:

$$|\exp(z)| = e^x.$$

- For $|\exp(2z+i)|$:

$$|\exp(2z+i)| = |\exp(2x + 2iy + i)| = e^{2x}|e^{i(2y+1)}| = e^{2x}.$$

- For $|\exp(iz^2)|$: Let $z^2 = (x+iy)^2 = x^2 - y^2 + 2ixy$,

$$|\exp(iz^2)| = |\exp(i(x^2 - y^2 + 2ixy))| = |e^{-2xy}e^{i(x^2-y^2)}| = e^{-2xy}.$$

Summing:

$$|\exp(2z+i)| + |\exp(iz^2)| = e^{2x} + e^{-2xy}.$$

Thus, the inequality is satisfied.

Problem 1(b): Show that $|\exp(z^2)| \leq \exp(|z|^2)$.

Solution:
Let $z = x + iy$, then:

$$z^2 = x^2 - y^2 + 2ixy.$$

Thus,

$$\exp(z^2) = e^{x^2-y^2}e^{i2xy}.$$

Taking the modulus:

$$|\exp(z^2)| = |e^{x^2-y^2}e^{i2xy}| = e^{x^2-y^2}.$$

Since $|z|^2 = x^2 + y^2$, we have:

$$e^{x^2-y^2} \leq e^{x^2+y^2}.$$

Thus:

$$|\exp(z^2)| \leq \exp(|z|^2).$$

Problem 1(c): Prove that $|\exp(-2z)| < 1$ if and only if $\mathrm{Re}(z) > 0$.

Solution:

$$\exp(-2z) = e^{-2x}e^{-i2y}.$$

Taking the modulus:

$$|\exp(-2z)| = |e^{-2x}e^{-i2y}| = e^{-2x}.$$

For $|\exp(-2z)| < 1$, we need:

$$e^{-2x} < 1.$$

Since e^{-2x} is always positive, this holds if and only if:

$$-2x < 0 \quad \Rightarrow \quad x > 0.$$

Thus, $|\exp(-2z)| < 1$ if and only if $\mathrm{Re}(z) > 0$.

Problem 2: Show that:

$$\overline{\exp(iz)} = \exp(i\overline{z})$$

if and only if:

$$z = n\pi, \quad n = 0, \pm 1, \pm 2, \ldots$$

Solution:

$$\exp(iz) = e^{i(x+iy)} = e^{ix}e^{-y}.$$

Taking the complex conjugate,

$$\overline{\exp(iz)} = e^{-ix}e^{-y}.$$

Similarly,

$$\exp(i\overline{z}) = e^{i(x-iy)} = e^{ix}e^{y}.$$

Thus, the given equation transforms into:

$$e^{-ix}e^{-y} = e^{ix}e^{y}.$$

Equating the magnitudes:

$$e^{-y} = e^{y} \quad \Rightarrow \quad -y = y \quad \Rightarrow \quad y = 0.$$

Thus, z must be purely real.

Equating the phase components:

$$e^{-ix} = e^{ix}.$$

This holds if and only if:

$$-ix = ix + 2\pi i n, \quad n \in \mathbb{Z}.$$

Solving for x:

$$-2ix = 2\pi i n \quad \Rightarrow \quad x = n\pi.$$

Thus, z must be of the form:

$$z = n\pi, \quad n \in \mathbb{Z}.$$

$$z = n\pi, \quad n = 0, \pm 1, \pm 2, \ldots$$

Problem 3(a): Compute $\frac{\partial F}{\partial \bar{z}}$ Using the Chain Rule

Given:

$$z = x + iy, \quad \bar{z} = x - iy,$$

and

$$x = \frac{z + \bar{z}}{2}, \quad y = \frac{z - \bar{z}}{2i}.$$

Applying the chain rule:

$$\frac{\partial F}{\partial \bar{z}} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial \bar{z}}.$$

We compute:

$$\frac{\partial x}{\partial \bar{z}} = \frac{1}{2}, \quad \frac{\partial y}{\partial \bar{z}} = \frac{i}{2}.$$

Thus,

$$\frac{\partial F}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial F}{\partial x} + i\frac{\partial F}{\partial y}\right).$$

Problem 3(b): Prove that $\frac{\partial f}{\partial \bar{z}} = 0$ Using the Cauchy-Riemann Equations

Define the operator:

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Let $f(z) = u(x, y) + iv(x, y)$, where u and v satisfy the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Applying the operator to f:

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left[\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) + i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\right].$$

Using the Cauchy-Riemann equations:

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left[(v_y - v_y) + i(v_x + u_y)\right] = 0.$$

Thus, we derive:

$$\frac{\partial f}{\partial \bar{z}} = 0.$$

This confirms analyticity.

Solution to the Problem on Complex Exponential Equality